# Assignment 1

① a) $\frac{dy}{dt} + \frac{1}{2}y = 2 + t$

$p(t) = \frac{1}{2}$ , $q(t) = 2 + t$

$$\mu(t) = e^{\int p(t)dt} = e^{\int \frac{1}{2}dt} = e^{\frac{1}{2}\int dt} = e^{\frac{1}{2}t} = e^{t/2}$$

$$e^{t/2}\left(\frac{dy}{dt} + \frac{1}{2}y\right) = e^{t/2}(2+t)$$

$$e^{t/2}\frac{dy}{dt} + \frac{e^{t/2}}{2}y = 2e^{t/2} + e^{t/2}\cdot t$$

$$\frac{d}{dt}\left(e^{t/2}y\right) = 2e^{t/2} + te^{t/2}$$

$$\int \frac{d}{dt}\left(e^{t/2}y\right) = \int (2e^{t/2} + te^{t/2})dt$$

$$e^{t/2}y = 2\int e^{t/2}dt + \int te^{t/2}dt$$

integration by parts.

$u = \frac{t}{2}$, $\frac{du}{dt} = \frac{1}{2}$, $dt = 2du$

$2\int e^u (2du)$, $4\int e^u du$, $4e^u$, $4e^{t/2}$

$$e^{t/2}y = 4e^{t/2} + \int \underbrace{t}_{u}\underbrace{e^{t/2}}_{dv}dt$$

$u = t$, $du = 1$

$dv = e^{t/2}$, $v = \int e^{t/2} = 2e^{t/2}$

$uv - \int v\,du$

$t(2e^{t/2}) - \int 2e^{t/2}(1)\,dt$

$2te^{t/2} - 4e^{t/2}$

$$e^{t/2}y = 4e^{t/2} + 2te^{t/2} - 4e^{t/2} + C$$

$$e^{t/2}y = 2te^{t/2} + C$$

$$y = 2t + \frac{C}{e^{t/2}}$$

$$\boxed{y = 2t + Ce^{-t/2}}$$

b) $y' - 2y = t^2e^{2t}$

$$\frac{dy}{dt} - 2y = t^2e^{2t}$$

$$p(t) = -2, \quad q(t) = t^2e^{2t}$$

$$u(t) = e^{\int p(t)\,dt} = e^{\int -2\,dt} = e^{-2\int dt} = e^{-2t}$$

$$e^{-2t}\left(\frac{dy}{dt} - 2y\right) = e^{-2t}t^2e^{2t}$$

$$e^{-2t}\frac{dy}{dt} - 2e^{-2t}y = t^2$$

$$\frac{d}{dt}\left(e^{-2t}y\right) = t^2$$

$$\int \frac{d}{dt}\left(e^{-2t}y\right) = \int t^2\,dt$$

$$e^{-2t}y = \frac{t^3}{3} + C$$

$$y = \frac{\frac{t^3}{3}}{e^{-2t}} + \frac{C}{e^{-2t}}$$

$$y = \frac{t^3}{3e^{-2t}} + \frac{c}{e^{-2t}}$$

$$\boxed{y = \frac{t^3 e^{2t}}{3} + ce^{2t}}$$

c) $x^3 \frac{dy}{dx} + 4x^2 y = e^{-x}$, $y(-1) = 0$

$$\frac{dy}{dx} + \frac{4x^2 y}{x^3} = \frac{e^{-x}}{x^3}$$

$$\frac{dy}{dx} + \frac{4}{x} y = \frac{e^{-x}}{x^3}, \quad p(x) = \frac{4}{x}, \quad q(x) = \frac{e^{-x}}{x^3}$$

$$u(x) = e^{\int p(x)dx} = e^{\int \frac{4}{x} dx} = e^{4\int \frac{1}{x} dx}$$

$$= e^{4\ln|x|} = e^{4\ln x}$$

$$= e^{\ln x^4}$$

$$= x^4.$$

$$x^4\left(\frac{dy}{dx} + \frac{4}{x} y\right) = x^4\left(\frac{e^{-x}}{x^3}\right)$$

$$x^4 \frac{dy}{dx} + 4x^3 y = x e^{-x}$$

$$\frac{d}{dx}(x^4 \cdot y) = x e^{-x}$$

(4)

$$\int \frac{d}{dx}(x^4 \cdot y) = \underbrace{\int x e^{-x} dx}_{\text{integration by parts}}$$

$u = x \quad dv = e^{-x}$

$du = 1 \quad v = \int e^{-x} = -e^{-x}$

$uv - \int v\,du$

$x(-e^{-x}) - \int -e^{-x}(1)dx$

$-xe^{-x} + \int e^{-x}dx$

$-xe^{-x} - e^{-x}$

$u = -x$

$\frac{du}{dx} = -1$

$-dx = du$

$dx = -du$

$\int e^u(-du)$

$-\int e^u$

$-e^u = -e^{-x}$

$$x^4 \cdot y = -xe^{-x} - e^{-x} + C$$

$$y = \frac{-xe^{-x}}{x^4} - \frac{e^{-x}}{x^4} + \frac{C}{x^4}$$

$$y = -\frac{e^{-x}}{x^3} - \frac{e^{-x}}{x^4} + \frac{C}{x^4}, \; y(-1) = 0$$

(2) a) $y' + \left(\frac{2}{t}\right)y = \frac{\cos t}{t^2}, \; y(\pi) = 0, \; t > 0$

$\frac{dy}{dt} + \left(\frac{2}{t}\right)y = \frac{\cos t}{t^2}, \; p(t) = \frac{2}{t}, \; q(t) = \frac{\cos t}{t^2}$

$\mu(t) = e^{\int p(t)dt} = e^{\int \frac{2}{t}dt} = e^{2\int \frac{1}{t}dt} = e^{2\ln|t|}$

$= e^{2\ln t}$, since $t > 0$

$= e^{\ln t^2} = t^2$

$$t^2\left(\frac{dy}{dt}+\left(\frac{2}{t}\right)y\right)=t^2\left(\frac{\cos t}{t^2}\right)$$

$$t^2\frac{dy}{dt}+t^2\left(\frac{2}{t}\right)y=\cos t$$

$$t^2\frac{dy}{dt}+2ty=\cos t$$

$$\frac{d}{dt}(t^2\cdot y)=\cos t$$

$$\int\frac{d}{dt}(t^2\cdot y)=\int\cos t$$

$$t^2y=\sin t+C$$

$$y=\frac{\sin t}{t^2}+\frac{C}{t^2}$$

$$y=\frac{\sin t}{t^2}+\frac{C}{t^2},\quad y(\pi)=0$$

$$0=\frac{\sin\pi}{\pi^2}+\frac{C}{\pi^2}$$

$$0=0+\frac{C}{\pi^2}\Rightarrow\frac{C}{\pi^2}=0$$

$$\Rightarrow C=0$$

$$\therefore\ y=\frac{\sin t}{t^2}+\frac{0}{t^2}$$

$$\boxed{y=\frac{\sin t}{t^2}}$$

(6)

b) $y' + \left(1 + \frac{2}{x}\right) y = \frac{\sin(x) e^{-x}}{x}, \quad x > 0, \quad y(\pi) = C$

$p(x) = e^{\int (1 + \frac{2}{x}) dx} = e^{x + 2\ln|x|}$

$p(x) = e^x \cdot e^{2\ln(x)} = e^x e^{\ln x^2} = x^2 e^x$

$x^2 e^x y' + x^2 e^x \left(1 + \frac{2}{x}\right) y = x^2 e^x \left(\frac{\sin x e^{-x}}{x}\right)$

$\frac{d}{dx}\left(x^2 e^x y\right) = x \sin x$

$x^2 e^x y = \int x \sin x \, dx \qquad uv - \int v \, du$

$= x(-\cos x) - \int -\cos x (1) dx \qquad u = x, \ du = 1$

$= -x\cos x + \int \cos x \, dx \qquad dv = \sin x, \ v = -\cos x$

$= -x\cos x + \sin x + C$

$y = -\frac{\cos x}{x e^x} + \frac{\sin x}{x^2 e^x} - \frac{\pi}{x^2 e^x}$

$y = -\frac{x\cos x}{x^2 e^x} + \frac{\sin x}{x^2 e^x} + \frac{C}{x^2 e^x}$

$y = -\frac{\cos x}{x e^x} + \frac{\sin x}{x^2 e^x} + \frac{C}{x^2 e^x}$

$y(\pi) = 0 \qquad 0 = \frac{-\cos\pi}{\pi e^\pi} + \frac{\sin\pi}{\pi^2 e^\pi} + \frac{C}{\pi^2 e^\pi}$

$0 = \frac{-(-1)}{\pi e^\pi} + \frac{C}{\pi^2 e^\pi} \qquad \frac{C}{\pi^2} + \frac{1}{\pi} = 0$

$0 = \frac{1}{e^\pi}\left(\frac{1}{\pi} + \frac{C}{\pi^2}\right) \qquad \frac{C}{\pi^2} = -\frac{1}{\pi}$

$y = -\frac{\cos x}{x e^x} + \frac{\sin x}{x^2 e^x} - \frac{\pi}{x^2 e^x} \qquad C = \frac{-\pi^2}{\pi} = -\pi$

(3) a) $\frac{dy}{dt} = \frac{te^t}{y\sqrt{1+y^2}}$ separable (7)

$$y\sqrt{1+y^2}\,dy = te^t\,dt$$

$$\int y\sqrt{1+y^2}\,dy = \int te^t\,dt$$

(I) (II)

look at each $\int$ seperate

(I) $\int y\sqrt{1+y^2}\,dy$, $u = 1+y^2$ $\quad 2y\,dy = du$

$\frac{du}{dy} = 2y$ $\quad y\,dy = \frac{du}{2}$

$$\int \sqrt{1+y^2}\; y\,dy$$

$$\int \sqrt{u}\left(\frac{du}{2}\right) = \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{2}\int u^{1/2}\,du$$

$$= \frac{1}{2}\,\frac{u^{3/2}}{\frac{3}{2}} + c_1$$

$$= \frac{1}{2}\cdot\frac{2}{3}\,u^{3/2} + c_1$$

$$= \frac{1}{3}u^{3/2} + c_1$$

$$= \frac{1}{3}(1+y^2)^{3/2} + c_1$$

(II) $\int te^t dt$ by parts $\quad u = t \quad dv = e^t$

$du = 1 \quad v = \int e^t = e^t$

$$uv - \int v\,du = te^t - \int e^t (1)\,dt$$

$$= te^t - e^t + c_2$$

combine answers back into DE expression

$$\frac{1}{3}(1+y^2)^{3/2} + c_1 = te^t - e^t + c_2$$

$$\frac{1}{3}(1+y^2)^{3/2} = te^t - e^t + c_2 - c_1$$

let $C = c_2 - c_1$

$$\frac{1}{3}(1+y^2)^{3/2} = te^t - e^t + C$$

$$\frac{1}{3}(1+y^2)^{3/2} = e^t(t-1) + C$$

$$(1+y^2)^{3/2} = 3e^t(t-1) + 3C, \quad K = 3C$$

$$(1+y^2)^{3/2} = 3e^t(t-1) + K$$

$$(1+y^2) = [3e^t(t-1) + K]^{2/3}$$

$$1 + y^2 = [3e^t(t-1) + K]^{2/3}$$

$$y^2 = [3e^t(t-1) + K]^{2/3} - 1$$

$$\boxed{y = \pm\sqrt{[3e^t(t-1) + K]^{2/3} - 1}}$$

b) $\frac{dy}{dx} = \frac{x^2+3y^2}{2xy}$, let $y = xv$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{x^2+3(xv)^2}{2x(xv)} = \frac{x^2+3x^2v^2}{2x^2v}$$

$$v + x\frac{dv}{dx} = \frac{x^2(1+3v^2)}{2x^2v} = \frac{(1+3v^2)}{2v}$$

$$x\frac{dv}{dx} = \frac{1+3v^2}{2v} - v = \frac{1+3v^2-2v^2}{2v}$$

$$x\frac{dv}{dx} = \frac{1+v^2}{2v} \Rightarrow \frac{dv}{dx} = \frac{1+v^2}{2xv} \Leftrightarrow$$

$$2xv\,dv = (1+v^2)\,dx$$

$$\frac{2v}{1+v^2}dv = \frac{dx}{x} \Rightarrow \int \frac{2v}{1+v^2}dv = \int \frac{dx}{x}$$

$u = 1+v^2$
$\frac{du}{v} = 2v$
$du = 2v\,dv$
$\int \frac{du}{u}$
$\ln|u|$
$\ln|1+v^2|$

$$\ln|1+v^2| = \ln|x| + C$$
$$\ln|1+v^2| - \ln|x| = C$$
$$y = xv \Rightarrow v = \frac{y}{x}$$
$$\ln\left|1+\left(\frac{y}{x}\right)^2\right| - \ln|x| = C$$

ln property

$$\ln\left[\frac{1+\left(\frac{y}{x}\right)^2}{x}\right] = C$$

$$\ln\left[\frac{1+\frac{y^2}{x^2}}{x}\right] = C$$

$$\ln\left[\frac{\frac{x^2+y^2}{x^2}}{x}\right] = C$$

$$\ln\left[\frac{x^2+y^2}{x^3}\right] = C$$

$$e^{\ln\left[\frac{x^2+y^2}{x^3}\right]} = e^C \text{, let } K = e^C$$

$$\frac{x^2+y^2}{x^3} = K \Rightarrow x^2+y^2 = Kx^3$$

$$\Rightarrow \boxed{x^2+y^2-Kx^3=0}$$

c) $\frac{dy}{dx} = \frac{x^2+xy+y^2}{x^2}$, let $y = xv$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{x^2+x(xv)+(xv)^2}{x^2}$$

$$v + x\frac{dv}{dx} = \frac{x^2+x^2v+x^2v^2}{x^2} = \frac{x^2(1+v+v^2)}{x^2}$$

$$v + x\frac{dv}{dx} = 1+v+v^2$$

$$x\frac{dv}{dx} = 1+v+v^2-v \Rightarrow x\frac{dv}{dx} = 1+v^2$$

$$\frac{dv}{dx} = \frac{1+v^2}{x} \quad \text{X}$$

$$x\,dv = (1+v^2)\,dx$$

$$\frac{dv}{1+v^2} = \frac{dx}{x} \Rightarrow \int \frac{dv}{1+v^2} = \int \frac{dx}{x}$$

$$\tan^{-1}(v) = \ln|x| + C$$
$$\tan^{-1}(v) - \ln|x| = C$$
$$y = xv \Rightarrow v = \frac{y}{x}$$

$$\boxed{\arctan(y/x) - \ln|x| = C}$$

d) $(x^2+1)\,y' = xy$

$(x^2+1)\frac{dy}{dx} = xy$ separable.

$(x^2+1)\,dy = xy\,dx$

$dy = \frac{x}{x^2+1}\,y\,dx$

$\frac{dy}{y} = \frac{x}{x^2+1}\,dx$

$\int \frac{dy}{y} = \int \frac{x}{x^2+1}\,dx$

for $\int \frac{x}{x^2+1}\,dx \Rightarrow \int \frac{\frac{du}{2}}{u}$

$u = x^2+1$

$\frac{du}{dx} = 2x$

$2x\,dx = du$

$x\,dx = \frac{du}{2}$

$\frac{1}{2}\int \frac{du}{u}$

$\frac{1}{2}\ln|u|$

$\Rightarrow \frac{1}{2}\ln|x^2+1|$

$$\ln|y| + c_1 = \frac{1}{2}\ln|x^2+1| + c_2$$

$$\ln y + c_1 = \frac{1}{2}\ln(x^2+1) + c_2$$

$$\ln y = \frac{1}{2}\ln(x^2+1) + (c_2 - c_1)$$

$$\ln y = \frac{1}{2}\ln(x^2+1) + C$$

$$\ln y = \ln(x^2+1)^{1/2} + C$$

ln property

$$e^{\ln y} = e^{[\ln(x^2+1)^{1/2} + C]}$$

$(e^{x+y} = e^x \cdot e^y)$

$$y = (x^2+1)^{1/2} \cdot e^C$$

Let $K = e^C$

$$y = (x^2+1)^{1/2} K$$

$$y = K(x^2+1)^{1/2}$$

$$\boxed{y = K\sqrt{x^2+1}}$$

(4) $y' = \dfrac{2\cos 2x}{(3+2y)}$, $y(0) = -1$

$$\frac{dy}{dx} = \frac{2\cos 2x}{3+2y}$$

$$(3+2y)\,dy = (2\cos 2x)\,dx$$

$$\int (3+2y)\,dy = \int (2\cos 2x)\,dx$$

$u = 2x$

$\dfrac{du}{dx} = 2$

$2\,dx = du$

$dx = \dfrac{du}{2}$

$$3y + \frac{2y^2}{2} + C_1 = 2\left(\frac{1}{2}\right)\sin 2x + C_2$$

$$3y + y^2 = \sin 2x + C, \quad y(0) = -1$$

$$-3 + 1 = \sin 0 + C \Rightarrow C = -2$$

$$\boxed{3y + y^2 = \sin 2x - 2}$$